

On the derivation of the spacetime metric from linear electrodynamics

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Abstract

In the framework of metric-free electrodynamics, we start with a *linear* spacetime relation between the excitation 2-form $H = (\mathcal{D}, \mathcal{H})$ and the field strength 2-form $F = (E, B)$. This linear relation is constrained by the so-called closure relation. We solve this system algebraically and extend a previous analysis such as to include also singular solutions. Using the recently derived fourth order *Fresnel* equation describing the propagation of electromagnetic waves in a general *linear* medium, we find that for all solutions the fourth order surface reduces to a light cone. Therefrom we derive the corresponding metric up to a conformal factor.

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1 Introduction

Recently, a metric-free formulation of the classical electromagnetic theory and its axiomatics has been discussed by Hehl and Obukhov, see refs. [1, 2]. The basic quantities in this formalism are the electromagnetic excitation 2-form H , directly related to the sources of the field, and the field strength 2-form F , describing the effects of the electromagnetic field on electrically charged test currents. The Lorentz force density $f_\alpha = (e_\alpha \rfloor F) \wedge J$ acts on test currents, where J is the electric current 3-form. The 2-forms H and F satisfy Maxwell's equations, $dH = J$, $dF = 0$. Field strength and excitations are not independent, but related by the so-called spacetime relation $H = H(F)$; see refs. [2, 3, 4] for details.

The spacetime relation, expressing the properties of the underlying spacetime, will in particular determine the dynamical properties of the electromagnetic field. In this way, it is the analogous of the constitutive law, determining the dynamical properties of the electromagnetic field *in material media*. The correspondence between electrodynamics in a gravitational field and in material media have been studied before, see for instance refs. [5, 6].

Here we consider the case of a *linear spacetime relation*, where H is proportional¹ to F . In terms of coordinate components, the field strength will be denoted as $F = (1/2) F_{ij} dx^i \wedge dx^j$ and the excitation 2-form as $H = (1/2) H_{ij} dx^i \wedge dx^j$. Then, the most general linear spacetime relation can be written in terms of the *electromagnetic spacetime tensor density* χ^{ijkl} (of weight +1), as²

$$H_{ij} = \frac{f}{4} \epsilon_{ijkl} \chi^{klmn} F_{mn}, \quad i, j = 0, 1, 2, 3, \quad (1)$$

where f is a dimensionfull scalar such that χ^{ijkl} is dimensionless. The spacetime tensor density has the following symmetries:

$$\chi^{ijkl} = -\chi^{jikl} = -\chi^{ijlk} = \chi^{klij}. \quad (2)$$

Note that the above formalism is metric-independent. As a particular example, however, one can recall the case of the coupling of the electromagnetic

¹Recently, some nonlinear models have been discussed in a similar formalism by Lorenci et al. [7, 8].

²We denote by ϵ_{ijkl} the completely antisymmetric tensor density of weight -1 with $\epsilon_{0123} := 1$.

field to gravity in Einstein's theory, i.e. minimal coupling to the metric tensor g_{ij} . This corresponds to the following particular electromagnetic spacetime tensor density [4],

$$\overset{\text{g}}{\chi}{}^{ijkl} := \sqrt{-g} (g^{ik} g^{jl} - g^{jk} g^{il}) , \quad (3)$$

which we do *not* assume to hold. Rather, we want to derive this equation from the linear ansatz (1).

2 Closure relation

With the spacetime tensor available, we may define a duality operator $\#$ acting on 2-forms by extending (1) to any 2-form $\Theta = \frac{1}{2} \Theta_{ij} dx^i \wedge dx^j$ such that

$$\# \Theta := \frac{1}{4} \epsilon_{ijkl} \chi^{klmn} \Theta_{mn} dx^i \wedge dx^j . \quad (4)$$

In refs. [1, 2, 9] spacetime tensors satisfying a so-called *closure relation* have been studied, namely those with

$$\#\# = -1 . \quad (5)$$

This additional condition can be motivated by considering the electric-magnetic reciprocity of the energy-momentum current³. Indeed, the closure relation results from assuming (1) to be electric-magnetic reciprocal, provided we choose f such that $f^2 \phi^2 = 1$. Note also that in the particular case (3) this condition is only fulfilled for metrics with Lorentzian signature.

In order to find the solutions of (5), it is convenient to adopt a more compact *bivector* notation by defining the indices $I, J, \dots = 01, 02, 03, 23, 31, 12$. In this notation, χ^{ijkl} corresponds to a symmetric 6×6 matrix χ^{IJ} and the totally antisymmetric ϵ -tensor density becomes

$$\epsilon^{IJ} = \epsilon_{IJ} = \begin{pmatrix} 0_3 & 1_3 \\ 1_3 & 0_3 \end{pmatrix} . \quad (6)$$

We define a new matrix κ by

$$\kappa_I{}^J := \epsilon_{IK} \chi^{KJ} . \quad (7)$$

³The energy-momentum 3-form of electrodynamics, $\Sigma_\alpha = (1/2) [F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)]$, is explicitly symmetric under a transformation $F \rightarrow \phi H$, $H \rightarrow -(1/\phi)F$, for an arbitrary pseudo scalar ϕ ; see also refs. [2, 10, 11] for a detailed discussion.

As $\epsilon^{IK} \kappa_K^J = \chi^{IJ}$, any solution of the closure relation is given by a real 6×6 matrix κ fulfilling

$$\kappa_I^K \kappa_K^J = -\delta_I^J, \quad (8)$$

provided $\epsilon^{IK} \kappa_K^J$ is symmetric.

We decompose the matrices χ^{IJ} and κ_I^J into 3×3 block-matrices,

$$\chi^{IJ} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad \kappa_I^J = \begin{pmatrix} C^T & B \\ A & C \end{pmatrix}, \quad (9)$$

with symmetric matrices A and B , and T denotes matrix transposition.

Then the closure relation, (5) or (8), translates into

$$C^2 + AB = -1_3, \quad (10)$$

$$BC + C^T B = 0_3, \quad (11)$$

$$CA + AC^T = 0_3. \quad (12)$$

A consideration of the following disjoint subcases will provide the general solution of the closure relation: 1) B regular, 2) B singular, but A regular, and 3) A and B singular.

3 Solutions of the closure relation

3.1 B regular

We construct the general solution of (10)-(12) for the case in which $\det B \neq 0$. Under these conditions (11) is solved by

$$C = B^{-1}K, \quad K^T = -K, \quad (13)$$

with an arbitrary antisymmetric matrix K . Using this solution for C , we rewrite (10) as

$$(B^{-1}K)^2 + AB = -1_3, \quad (14)$$

so that the solution for A is given by

$$A = -B^{-1} [1 + (KB^{-1})^2]. \quad (15)$$

The symmetry of A , as assumed in (9), may easily be seen in eq. (18) below.

A short calculation shows that the solutions (13) and (15) satisfy also (12) identically:

$$\begin{aligned}
CA + AC^T &= (B^{-1}K) \left[-B^{-1}(1 + (KB^{-1})^2) \right] \\
&\quad + \left[-B^{-1}(1 + (KB^{-1})^2) \right] (-KB^{-1}) \\
&= -B^{-1}KB^{-1} - B^{-1}KB^{-1}(KB^{-1})^2 \\
&\quad + B^{-1}KB^{-1} + B^{-1}(KB^{-1})^2KB^{-1} \\
&= 0.
\end{aligned} \tag{16}$$

Therefore, the solution of the closure relation can be written as

$$\chi^{IJ} = \begin{pmatrix} -B^{-1}[1 + (KB^{-1})^2] & B^{-1}K \\ -KB^{-1} & B \end{pmatrix}, \tag{17}$$

provided $\det B \neq 0$. This solution consists of 9 independent parameters: 3 from the antisymmetric matrix K and 6 from the nonsingular symmetric matrix B .

The solution previously found in ref. [9], eq. (14), is equivalent to ours, as a simple calculation using computer algebra shows

$$A = -B^{-1}(1 + (KB^{-1})^2) = pB^{-1} + qN, \tag{18}$$

with $K = (K_{ab}) = (\epsilon_{abc}k^c)$, $N = (N^{ab}) = (k^ak^b)$, $q = -1/\det B$, and $p = -1 + \text{tr}(NB)/\det B$.

3.2 B singular, A regular

In this case (17) is not valid, but one can find a solution following a similar procedure, now starting with an arbitrary non-singular matrix A . We solve (12) with respect to C and then use that result to solve (10) with respect to B . In this way the solution can be found to be

$$\chi^{IJ} = \begin{pmatrix} A & LA^{-1} \\ -A^{-1}L & -A^{-1}[1 + (LA^{-1})^2] \end{pmatrix}. \tag{19}$$

Here L is again an arbitrary antisymmetric matrix, with components $L = (L^{ab})$, $L^{ab} =: \epsilon^{abc}l_c$.

This solution has 8 independent parameters: 3 from the antisymmetric matrix L and 6 from the nonsingular symmetric matrix A , fulfilling the constraint⁴ $\det B = 0$. The constraint reads explicitly

$$\det B = -\det \left(A^{-1} \left[1 + (LA^{-1})^2 \right] \right) = 0, \quad (20)$$

or, equivalently, $\det(A) - A^{ab}l_al_b = 0$, as one finds after some algebra.

3.3 A and B singular

Finally, we analyse the case in which both $\det B = 0$ *and* $\det A = 0$. For simplicity, we will work in the basis in which the symmetric matrix B is diagonal. Since B is singular, we can choose the basis such that $B_0 = \text{diag}(0, b_{22}, b_{33})$. After inserting this ansatz into equations (10)-(12), one finds that at least one of the two eigenvalues of B must vanish, otherwise there is no real solution. Furthermore, at least one of the eigenvalues of B must be different from zero. Otherwise, if $B = 0$, eq. (10) would imply $C^2 = -1_3$ which has no solution with a real 3×3 matrix. Then, we choose the basis such that $B_0 = \text{diag}(0, 0, b_{33})$, with $b_{33} \neq 0$. We denote the solution in this particular basis as A_0, B_0 and C_0 . Using this form of the matrix B , one is able to find, after some algebra, that the equations (10)-(12) admit the following *four parameter* solution:

$$A_0 = \begin{pmatrix} -c_{23}^2 c_{12}^2 b_{33}^{-1} & c_{13} c_{23} b_{33}^{-1} & -c_{12} c_{23} b_{33}^{-1} \\ c_{13} c_{23} b_{33}^{-1} & -c_{13}^2 c_{12}^{-2} b_{33}^{-1} & c_{13} c_{12}^{-1} b_{33}^{-1} \\ -c_{12} c_{23} b_{33}^{-1} & c_{13} c_{12}^{-1} b_{33}^{-1} & -b_{33}^{-1} \end{pmatrix}, \quad (21)$$

$$B_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_{33} \end{pmatrix}, \quad (22)$$

$$C_0 = \begin{pmatrix} 0 & c_{12} & c_{13} \\ -c_{12}^{-1} & 0 & c_{23} \\ 0 & 0 & 0 \end{pmatrix}. \quad (23)$$

No real solution exists if $c_{12} = 0$. The general solution can thus be found from this special one by a similarity transformation:

$$A = S A_0 S^{-1} \quad B = S B_0 S^{-1} \quad C = S C_0 S^{-1}. \quad (24)$$

⁴Note, this solution is also valid for the case $\det A \neq 0$ *and* $\det B \neq 0$. But in this case, (19) is just a reparametrization of (17).

Here S is an arbitrary regular matrix which keeps A and B symmetric. This means that S has to be orthogonal (see for instance ref. [12], p. 297). Therefore, the general solution has 7 independent components, due to the 3 additional parameters corresponding to the orthogonal transformation. These 7 independent components are thus in agreement with the 9 independent components of the regular solution, provided one takes into account the two conditions $\det A = \det B = 0$.

The orthogonal group $O(3)$ is given by the direct product of parity transformations P and the special orthogonal subgroup $SO(3)$. We just have to consider $SO(3)$ transformations as parity transformations leave the solution invariant. We consider the parametrization of $SO(3)$ based on the generators of rotations with respect to the Cartesian axes,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \text{and} \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (25)$$

Any matrix $S \in SO(3)$ is given by

$$S(\theta^a) = \exp(iJ_a\theta^a). \quad (26)$$

Hence, the general solution for this case is given by

$$A = \exp(iJ_a\theta^a) A_0 \exp(-iJ_a\theta^a), \quad (27)$$

$$B = \exp(iJ_a\theta^a) B_0 \exp(-iJ_a\theta^a), \quad (28)$$

$$C = \exp(iJ_a\theta^a) C_0 \exp(-iJ_a\theta^a). \quad (29)$$

4 The extraction of the metric

4.1 The regular solution of the closure relation

For a regular matrix B , the metric has been extracted in two independent ways up to a conformal factor. The first method [1, 9] makes use of two formulas by Urbantke, see refs. [13, 14]. On the other hand, Obukhov et al. [15] have studied the propagation of electromagnetic waves for the most general case of a linear constitutive relation, see (1). They found that the wave covector, q_i , describing the propagation of wave fronts, fulfills a fourth order equation

$$G^{ijkl} q_i q_j q_k q_l = 0, \quad (30)$$

with

$$G^{ijkl} := \frac{1}{4!} \chi^{mnp(i} \chi^{j|qr|k} \chi^{l)stu} \epsilon_{mnrs} \epsilon_{pqtu}. \quad (31)$$

This fourth order Fresnel equation is found to reduce to an equation of second order in the case of the regular solution (17) of the closure relation (5), i.e.

$$G^{ijkl} q_i q_j q_k q_l = \sqrt{|g|} (g^{ij} q_i q_j)^2. \quad (32)$$

The factor $\sqrt{|g|}$, with $g := \det(g_{ij})$, is necessary since χ^{ijkl} is a tensor density of weight +1 so that G^{ijkl} is also a tensor density of weight +1.

As $g^{ij} q_i q_j = 0$ defines the lightcone at each event, one can read off the metric coefficients from (32), up to a conformal factor. In this case, the resulting metric, obtained by the two different methods, reads

$$g^{ij} = \frac{1}{\sqrt{\det B}} \left(\frac{1 - (\det B)^{-1} k_c k^c}{-k^a} \middle| \frac{-k^b}{-(\det B)(B^{-1})^{ab}} \right), \quad (33)$$

with $k_a := B_{ab} k^b$. It can be shown that g^{ij} has Lorentzian signature. For details see refs. [1, 9].

4.2 Solution for regular A and singular B

After evaluating (30) and (31) for the solution (19), we find that the Fresnel equation (30) also separates. From it, we read off the components of the metric up to a conformal factor. We find

$$g^{ij} = \left(\frac{\det A}{l^a} \middle| \frac{l^b}{(\det A)^{-1} l^a l^b - A^{ab}} \right), \quad (34)$$

where we have defined $l^c := A^{ac} l_a$. Again, this metric has Lorentzian signature, since $g := \det(g_{ij}) = -\det A^{-2} < 0$.

4.3 The degenerated case

Finally, for our special solution (21)-(23), we find that (30) also separates. The corresponding metric, up to a conformal factor, is found to be

$$g_0^{ij} = \begin{pmatrix} 0 & c_{23} c_{12}^2 & -c_{13} & c_{12} \\ c_{23} c_{12}^2 & b_{33} c_{12}^2 & 0 & 0 \\ -c_{13} & 0 & b_{33} & 0 \\ c_{12} & 0 & 0 & 0 \end{pmatrix}. \quad (35)$$

The determinant in this case is $g = -b_{33}^{-2}c_{12}^{-4} < 0$, so that the metric has Lorentzian signature, as in the previous cases.

In order to study the case of solution (24), we look how the quantities, in particular G^{ijkl} as defined in (30), change under the orthogonal transformation⁵ S .

From (9) and (24) one can write the corresponding transformed spacetime tensor density as

$$\chi^{ijkl} = \Lambda^i_p \Lambda^j_q \Lambda^k_r \Lambda^l_s \chi_0^{pqrs}, \quad (36)$$

where we have defined

$$\Lambda^i_j := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & S \end{array} \right). \quad (37)$$

From (31) we find the corresponding expression for the transformed tensor density G^{ijkl} . Since $\det(\Lambda^i_j) = 1$, it takes the simple form

$$G^{ijkl} = \Lambda^i_p \Lambda^j_q \Lambda^k_r \Lambda^l_s G_0^{pqrs}. \quad (38)$$

Using this transformation law, one can easily prove the following: For orthogonal transformations (24) and (37), the tensor density G^{ijkl} reduces the forth order Fresnel equation to the light cone equation, see (32), provided the tensor density G_0^{ijkl} does. Thus, we find that the resulting transformed metric can be written in terms of the initial one in the form:

$$g^{ij} = \Lambda^i_k \Lambda^j_l g_0^{kl}. \quad (39)$$

We do not give here the explicit expression for this metric. We observe however that it has 7 independent components, and a Lorentzian signature.

5 Conclusions

By completing a recent analysis, we found a 1-to-1 correspondence between spacetime tensors fulfilling the closure relation and the conformally invariant part of metric tensors with Lorentzian signature. A new, more instructive proof for the regular solution (17) is given.

In all cases, we checked the following important property: We took the conventional Hodge dual of a 2-form with respect to the metric we derived.

⁵Alternatively, one can consider transformations of the coframe basis, as pointed out by Obukhov [2, 16].

This coincides with the action of the duality operator $\#$, defined in terms of the spacetime relation, fulfilling the closure relation. In other words, we have derived (3) from our linear ansatz (1), the latter of which is constrained by the closure relation (5).

This result, previously found only for regular sub-matrices A and B , is then valid for all solutions.

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